A reduction of the Navarro Alperin weight conjecture and its blockwise version

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(Joint work with Z. Feng and Y. Zhou)

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In 2004, Navarro refined the Alperin weight conjecture by incorporating Galois automorphisms. The resulting conjecture, called the *Navarro–Alperin weight (NAW) conjecture*, asserts that the bijection predicted by the Alperin weight conjecture can be chosen to be equivariant under the action of Galois automorphisms.

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In 2011, Navarro and Tiep reduced the Alperin weight conjecture to finite quasi-simple groups. They formulated the *inductive Alperin weight condition* and proved that, if this inductive condition holds for all finite quasi-simple groups, then the Alperin weight conjecture is valid for all finite groups.

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There exists a series of reductions of the so-called global–local conjectures (including the Alperin weight conjecture) in the literature, carried out by Isaacs, Malle, Späth, and others.

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In this talk, we introduce these concepts and the related key results, and then refine them by considering Galois automorphisms.

Character triples, projective representations and factor sets.

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$$\mathcal{P} : G \to \mathrm{M}_l(F)^\times$$

satisfying (i) $\mathrm{Res}_N^G(\mathcal{P})$ affords θ,
(ii) $\mathcal{P}(g)\mathcal{P}(n) = \mathcal{P}(gn)$ and $\mathcal{P}(n)\mathcal{P}(g) = \mathcal{P}(ng)$, $n \in N, g \in G$.

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Factor set of $\mathcal{P}$: The map $\alpha : G \times G \to F^\times$ satisfying

$$\mathcal{P}(g)\mathcal{P}(h) = \alpha(g, h)\mathcal{P}(gh), \quad \forall g, h \in G.$$

It descents to $\bar{\alpha} : G/N \times G/N \to F^\times$.

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If the factor sets $\bar{\alpha}$ of $\mathcal{P}$ and $\bar{\alpha}'$ of $\mathcal{P}'$ coincide via ϵ, then for every middle subgroup $N \leq \mathcal{J} \leq G$, $(\mathrm{Res}^G_{\mathcal{J}}\mathcal{P}, \mathrm{Res}^H_{\mathcal{J}_1}\mathcal{P}')$ determines a bijection

$$\nu_{\mathcal{J}} : \mathrm{IBr}(\mathcal{J} \,|\, \theta) \to \mathrm{IBr}(\mathcal{J}_1 \,|\, \varphi), \text{ where } \epsilon(\mathcal{J}/N) = \mathcal{J}_1/M.$$

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If this happens, we say that $\nu := \{\nu_{\mathcal{J}} \,|\, N \leq \mathcal{J} \leq G\}$ is the isomorphism between these two character triples, and it is *given* by $(\mathcal{P}, \mathcal{P}')$.

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Let (G, N, θ) be a character triple. Then there exists another character triple (H, M, φ), with M central in H, together with an isomorphism ν between them.

Galois automorphisms and $\mathcal{H}$-triples

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$(\mathcal{K}, \mathcal{O}, F)$: p-modular system
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$(G, N, \theta)_{\mathcal{H}}$: (Modular) $\mathcal{H}$-triple, consists of finite groups $N \trianglelefteq G$, and $\theta \in \mathrm{IBr}(N)$ such that $G = G_{\theta^{\mathcal{H}}}$, where $\theta^{\mathcal{H}}$ is the $\mathcal{H}$-orbit of θ and $G_{\theta^{\mathcal{H}}}$ is the stabilizer of $\theta^{\mathcal{H}}$ in G.

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Note that G_θ, the stabilizer of θ in G, is normal in G, and (G_θ, N, θ) is a character triple.

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E-realized rep. X of θ: A rep. $X : N \to \mathrm{M}_l(F)^\times$ affording θ, with every $X(n)$ contained in $M_l(E)^\times$ (regard $\mathrm{M}_l(E) \subseteq \mathrm{M}_l(F)$).
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Double centralizer theorem: $\mathrm{M}_l(E)$ and E are both centralizer of each other in $\mathrm{M}_{ls}(F_p)$.

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A proj. rep. associated to $(G, N, \theta)_{\mathcal{H}}$ is a map

$$\mathcal{P} : G \to \mathrm{M}_{s\theta(1)}(F_p)^{\times}$$

satisfying (i). $\mathcal{P}(n) \in \mathrm{M}_{\theta(1)}(E), \forall n \in N$, and $\mathrm{Res}^G_N(\mathcal{P})$ is a E-realized rep. of N affording θ.

(ii) $\mathcal{P}(g)\mathcal{P}(n) = \mathcal{P}(gn), \mathcal{P}(n)\mathcal{P}(g) = \mathcal{P}(ng), g \in G, n \in N.$

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The double centralizer theorem guarantees that $\mathcal{P}(g)$ is unique up to multiplication by an element of $E^{\times}$. Thus, for every $g, h \in G$, there exists $\alpha(g, h) \in E^{\times}$ such that $\mathcal{P}(g)\mathcal{P}(h) = \mathcal{P}(gh)\alpha(g, h)$.

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The map $\alpha : G \times G \to E^{\times}$ is the factor set of $\mathcal{P}$.

Projective representations associated to $\mathcal{H}$-triples

Indeed, $\alpha \in \mathbf{Z}^2(G, E^\times)$, where $g \in G$ acts on $E^\times$ via the automorphism $\sigma \in \mathcal{H}$ such that (σ, g) stabilizes θ. Recall that $\theta^{(\sigma,g)}(n) = \theta(gng^{-1})^\sigma$. This action is well-defined.

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α descents to an $\bar{\alpha} \in \mathbf{Z}^2(G/N, E^\times)$.

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(i). $(\mathcal{H} \times G/N)_\theta = (\mathcal{H} \times H/M)_\varphi$, via ϵ.
This implies that $G_\theta/N = H_\varphi/M$, and that $F_p[\theta] = F_p[\varphi]$ by $\mathcal{H}_\theta = \mathcal{H}_\varphi$ (fundamental Galois theorem).

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(ii). There exist proj. rep. $\mathcal{P}, \mathcal{P}'$ associated to $(G, N, \theta)_{\mathcal{H}}, (H, M, \varphi)_{\mathcal{H}}$ with factor sets $\bar{\alpha}, \bar{\alpha}'$, resp., such that $\bar{\alpha} = \bar{\alpha}'$ via ϵ.

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If this happens, then for every middle subgroup $N \leq \mathcal{J} \leq G_\theta$, the bijection

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determined by $(\mathrm{Res}^G_{\mathcal{J}} \mathcal{P}, \mathrm{Res}^H_{\mathcal{J}_1} \mathcal{P}')$, is $(\mathcal{H} \times \mathbf{N}_{G/N}(\mathcal{J}/N))_\theta$-equivariant.

Reduction of an $\mathcal{H}$-triple to one with central subgroup

A key result is:
Any $\mathcal{H}$-triple $(G, N, \theta)_{\mathcal{H}}$ is isomorphic to one $(H, M, \varphi)_{\mathcal{H}}$ with M central in H_φ.

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(iii). Set $\widehat{N} := \{(n, z) \mid n \in N, z \in F_p[\theta]^\times\}$ and $N_0 := \{(n, 1) \mid n \in N\}$. Both are normal in $\widehat{G}$.

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(iv). Set $H := \widehat{G}/N_0$ and $M := \widehat{N}/N_0$, and φ is induced by the identification $M \cong F_p[\theta]^{\times}$ (or inverse).
(The procedure is similar with the case of character triples.)

Remaining questions

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(ii). Can the result (i), if established, be useful for the reduction of the Alperin–McKay–Navarro conjecture? The latter still remains open.

Thanks for Listening